

The L^1 Liouville Property on Weighted manifolds

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ABSTRACT. We consider a complete noncompact smooth metric measure space $(M^n, g, e^{-f} dv)$ and the associated drifting Laplacian. We find sufficient conditions on the geometry of the space so that every nonnegative f -subharmonic function with bounded weighted L^1 norm is constant.

1. Introduction

We consider a complete noncompact smooth metric measure space

$$(M^n, g, e^{-f} dv),$$

where (M^n, g) is a complete noncompact smooth Riemannian manifold with a weighted volume measure $d\mu = e^{-f} dv$ such that f is a smooth function on M and dv is the Riemannian measure. In this paper, we refer to such a space as a *weighted manifold*.

The associated drifting Laplacian on such a weighted manifold is

$$\Delta_f = \Delta - \nabla f \cdot \nabla,$$

where Δ is the Laplace operator and ∇ is the gradient operator on the Riemannian manifold M . Δ_f can be extended to a densely defined, self-adjoint, nonpositive definite operator on the space of square integrable functions with respect to the measure $d\mu$. A smooth function u is called *f-harmonic*, whenever

$$\Delta_f u = 0$$

and f -subharmonic if

$$\Delta_f u \geq 0.$$

We use $\langle \cdot, \cdot \rangle$ to denote the inner product of the Riemannian metric and $|\cdot|$ to denote the corresponding norm. The weighted L^p norms are defined as

$$\|u\|_p = \left(\int_M |u|^p e^{-f} dv \right)^{1/p}$$

for any $1 \leq p < \infty$. The weighted L^p space is then given by

$$L^p(M, e^{-f} dv) = \{u \mid \|u\|_p < \infty\}.$$

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We say that the weighted manifold satisfies the L^1 *Liouville property* if every nonnegative f -subharmonic function that belongs to L^1 is constant. In this article, we are interested in finding sharp conditions on the curvature of the weighted manifold and f so that it satisfies the L^1 Liouville property.

In the classical case $f = 0$, P. Li demonstrated that a manifold satisfies the L^1 Liouville property whenever its Ricci curvature is bounded below by a negative quadratic function: $\text{Ric}(x) \geq -C(1 + r(x)^2)$, where $r(x) = d(x, x_o)$ is the distance to a fixed point x_o [4]. This is achieved by proving an appropriate mean value inequality for subharmonic functions and finding adequate heat kernel bounds on geodesic balls at x_o . The main ingredients in the proof of both the mean value inequality and the heat kernel bound were the Bochner formula, the Laplacian comparison theorem and volume comparison for general geodesic balls. The lower bound on the Ricci curvature is sharp as is shown in [6].

In the case of weighted manifolds, it was shown by Bakry and Émery that the analogous Bochner formula can be obtained if one takes as the curvature tensor

$$\text{Ric}_f = \text{Ric} + \nabla^2 f$$

where Ric is the Ricci curvature of the Riemannian manifold and $\nabla^2 f$ is the Hessian of the function f [1]. Ric_f is known as the Bakry-Émery Ricci curvature. However, the Bochner formula for the drifting Laplacian remains slightly different. It is given by

$$(1) \quad \Delta_f |\nabla u|^2 = 2|\nabla^2 u|^2 + 2\langle \nabla u, \nabla \Delta_f u \rangle + 2\text{Ric}_f(\nabla u, \nabla u),$$

where $\nabla^2 u$ is the Hessian of u and $|\nabla^2 u|^2 = \sum u_{ij}^2$. We do not have a notion of f -Hessian and what appears in the above Bochner formula for the drifting Laplacian is the usual Hessian. Observe that when $f = 0$, (1) becomes the Bochner formula in the Riemannian case.

It was previously shown that the weighted manifold satisfies the L^1 Liouville property whenever $\text{Ric}_f \geq 0$ and f has quadratic growth [13], and also whenever $\text{Ric}_f(x) \geq -K(1 + r(x)^2)$ and f is bounded [12]. In this article, using an alternative method from the one in [12, 13], we will prove the following more general result

THEOREM 1.1. *Let $(M^n, g, d\mu)$ be a complete metric measure space with $d\mu = e^{-f} dv$ and fix a point $x_o \in M$. Suppose that for all $R > 1$ the manifold satisfies the weighted volume form condition (V_R) as in Definition 2.1 with $A(R) = bR^2$ and some uniform constants a, b that do not depend on R .*

Then every nonnegative f -subharmonic function in L^1 must be constant.

As we will see, our result indicates that the weighted manifold will satisfy the L^1 Liouville property under various conditions on the Bakry-Émery Ricci tensor and f that guarantee the following basic principle: geodesic balls of radius r have weighted volume growth of the order e^{cR^2} , and the mean value inequalities and heat kernel estimates also hold on geodesic balls with the additional factor of e^{cR^2} . In Section 2 we obtain a mean value inequality for f -subharmonic functions, as well Gaussian estimates for the drifting heat kernel under a more general volume form condition. We provide the proof of Theorem 1.1 in Section 3. In Section 4 we give conditions on Ric_f and f that guarantee the assumptions of the theorem and in the final section we show that the above conditions are sharp.

2. Mean Value Inequality and Heat Kernel Estimate

We fix a point $x_o \in M$ and set $r(x) = d(x, x_o)$ the radial distance to x_o . We denote the geodesic ball of radius R at x by $B_x(R)$, and its volume in the weighted measure by $V_f(x, R)$.

Using the classical Li-Yau method, one can obtain gradient estimates and mean value inequalities for subharmonic functions whenever $\Delta r(x, y) \leq \frac{a}{r(x, y)} + b$ for some uniform constants a, b (here $r(x, y) = d(x, y)$). Such uniform Laplacian comparison theorem is easy to obtain in the case $f = 0$, by simply assuming that the Ricci curvature of the manifold is bounded below. In the case of weighted manifolds however, $\text{Ric}_f(x) \geq -(n-1)K$ on a ball around x_o does not imply a uniform Laplacian estimate $\Delta_f r(x, y) \leq C \frac{1}{r(x, y)} + b$ without strong restrictions on f .

In this article we will use an alternative technique based on the method of Saloff-Coste [10]. As in the case $f = 0$, we can prove a local Neumann Poincaré inequality for smooth functions in $(M^n, g, d\mu)$, whenever we can compare the volume form of the weighted manifold to the volume form of an a -dimensional manifold with Ricci curvature bounded below. This is done by following the proof of Theorem 5.6.5 in [10]. Furthermore, we can prove a local Sobolev inequality in a similar manner. The Sobolev inequality then implies a mean value inequality for f -subharmonic functions as well as a mean value inequality for solutions to the drifting heat equation. We use these to prove a Gaussian estimate for the drifting heat kernel.

For any point $x \in M$ we denote the Riemannian volume form in geodesic coordinates at x by

$$dv(\exp_x(r\xi)) = J(x, r, \xi) dr d\xi$$

for $r > 0$ and ξ any unit tangent vector at x . Then the f -volume form in geodesic coordinates is given by

$$J_f(x, r, \xi) = e^{-f} J(x, r, \xi).$$

If $y = \exp_x(r\xi)$ is a point that does not belong to the cut-locus of x , then

$$\Delta r(x, y) = \frac{J'(x, r, \xi)}{J(x, r, \xi)} \quad \text{and} \quad \Delta_f r(x, y) = \frac{J'_f(x, r, \xi)}{J_f(x, r, \xi)}$$

where $r(x, y) = d(x, y)$ and the derivatives are taken in the radial direction. The first equality is the key element in Bishop's volume comparison theorem under the assumption of a uniform Laplacian upper bound. Analogously, on weighted manifolds, the second equality provides us with weighted volume comparison results whenever we have a uniform drifting Laplacian upper bound. We will be showing that it is in fact sufficient to have a uniform volume form comparison assumption.

DEFINITION 2.1. *We say that the smooth metric measure space $(M^n, g, d\mu)$ satisfies the property (V_R) , if there exists a positive and nondecreasing function $A(R)$ and a uniform constant a (independent of R) such that for all $x \in B_{x_o}(R)$ and $0 < r_1 < r_2 < R$*

$$(V_R) \quad \frac{J_f(x, r_2, \xi)}{J_f(x, r_1, \xi)} \leq \left(\frac{r_2}{r_1} \right)^a e^{A(R)}.$$

The above inequality is assumed for all points $\exp_x(r_i \xi)$ that do not belong to the cut locus of x .

The classical integration argument in Bishop's theorem gives us the following volume comparison result

LEMMA 2.2. *Suppose that $(M^n, g, d\mu)$ satisfies the property (V_R) for all $x \in B_{x_o}(R)$. Then for any $0 < r_1 < r_2 < R$*

$$\frac{V_f(x, r_2)}{V_f(x, r_1)} \leq \left(\frac{r_2}{r_1}\right)^{a+1} e^{A(R)}.$$

As mentioned above, the proof in [10, Theorem 5.6.5] illustrates that assumption (V_R) , and the consequent volume comparison result that follows from it, are enough to prove a local Neumann Poincaré inequality on the weighted manifold (see also the proof in [8, Lemma 3.4] and [13, Lemma 2.3], and [12, Lemma 2.4]). The local Sobolev inequality then follows as in [11, Theorem 2.1]. We summarize these results below

LEMMA 2.3. *Let $(M^n, g, d\mu)$ be a metric measure space that satisfies the property (V_R) for all $x \in B_{x_o}(R)$. Then for any $x \in B_{x_o}(R)$, $0 < r < R$ and $\varphi \in \mathcal{C}^\infty(B_x(r))$*

$$\int_{B_x(r)} |\varphi - \varphi_{B_x(r)}|^2 d\mu \leq C_1 e^{C_2 A(R)} r^2 \int_{B_x(r)} |\nabla \varphi|^2 d\mu$$

where $C_1 = C_1(n, a)$, $C_2 = C_2(n)$ and $\varphi_{B_x(r)} = V_f(x, r)^{-1} \int_{B_x(r)} \varphi d\mu$.

Furthermore, there exist constants $\nu = \nu(n) > 2$, $C_3(n, a)$ and $C_4(n)$ such that for all $\varphi \in \mathcal{C}_0^\infty(B_x(r))$

$$(2) \quad \left(\int_{B_x(r)} |\varphi|^{\frac{2\nu}{\nu-2}} d\mu \right)^{\frac{\nu-2}{\nu}} \leq C_3 \frac{e^{C_4 A(R)} r^2}{V_f(x, r)^{\frac{2}{\nu}}} \int_{B_x(r)} (|\nabla \varphi|^2 + r^{-2} |\varphi|^2) d\mu.$$

As in the articles mentioned above, we can now use the Sobolev inequality to prove a local mean value inequality for positive solutions of the drifting heat equation [11, Theorem 3.1], [13, Proposition 2.6].

LEMMA 2.4. *Let $(M^n, g, d\mu)$ be a smooth metric measure space that satisfies the local Sobolev inequality (2) of Lemma 2.3 for all $\varphi \in \mathcal{C}_0^\infty(B_{x_o}(\rho))$ and $0 < \rho < R$. Let u be a smooth positive solution of the drifting heat equation*

$$\partial_t u - \Delta_f u = 0$$

in the cylinder $Q = B_{x_o}(r) \times (s - r^2, s)$ for some $s \geq 0$ and $r < R$. Then for any $0 < \delta < \delta' \leq 1$ there exist constants $C_5(n, a, \nu)$, $C_6(n, \nu)$ such that

$$\sup_{Q_\delta} u \leq C_5 \frac{e^{C_6 A(R)}}{(\delta' - \delta)^{2+\nu} r^2 V_f(x_o, r)} \int_{Q_{\delta'}} u d\mu dt$$

where $Q_\alpha = B_{x_o}(\alpha r) \times (s - \alpha r^2, s)$.

Moreover, we obtain a mean value inequality for subsolutions to the drifting heat equation as in [10, Theorem 5.2.9]

LEMMA 2.5. *Let $(M^n, g, d\mu)$ be a smooth metric measure space that satisfies the local Sobolev inequality (2) of Lemma 2.3 for all $\varphi \in \mathcal{C}_0^\infty(B_{x_o}(\rho))$ and $0 < \rho \leq R$. Fix $0 < p < \infty$ and let v be a positive subsolution of the drifting heat equation such that*

$$\partial_t v - \Delta_f v \leq 0$$

in the cylinder $Q = B_{x_o}(r) \times (s - r^2, s)$ for some $s \geq 0$ and $r < R$. Then for any $0 < \delta < \delta' \leq 1$ there exist constants $C_7(n, a, \nu, p), C_8(n, \nu, p)$ such that

$$\sup_{Q_\delta} u^p \leq C_7 \frac{e^{C_8 A(R)}}{(\delta' - \delta)^{2+\nu} r^2 V_f(x_o, r)} \int_{Q_{\delta'}} u^p d\mu dt.$$

The following corollary is immediate by setting $p = 1$ and observing that an f -subharmonic function is also a subsolution to the drifting heat equation

COROLLARY 2.1. *Let $(M^n, g, d\mu)$ be a smooth metric measure space as in Lemma 2.5. Let v be a positive f -subharmonic function on M . Then there exist constants $C_9(n, a), C_{10}(n)$ such that*

$$\sup_{B_{x_o}(\frac{R}{2})} u \leq C_9 \frac{e^{C_{10} A(R)}}{R^2 V_f(x_o, R)} \int_{B_{x_o}(R)} u d\mu.$$

The bottom of the Rayleigh quotient on the weighted manifold is defined as

$$\lambda_1(M) = \inf_{g \in \mathcal{C}_c^\infty(M)} \frac{\int_M |\nabla g|^2 d\mu}{\int_M |g|^2 d\mu}.$$

Whenever f has linear growth rate at a point, Munteanu and Wang prove an upper bound for $\lambda_1(M)$ in the case $\text{Ric}_f \geq 0$ [8] and a sharp upper upper bound when $\text{Ric}_f \geq -(n-1)$ [9].

For any compact domain, $\Omega \subset M$, the drifting Laplacian Δ_f with Dirichlet boundary conditions is a nonpositive, densely defined and self-adjoint operator on $L^2(\Omega, e^{-f} dv)$. Moreover, the first eigenvalue of the spectrum of Δ_f for the Dirichlet problem satisfies $\lambda_1(\Omega) > 0$.

We also make the following observation: Let $\{\Omega_i\}$ be a sequence of compact sets such that the boundary of each Ω_i is piecewise smooth, $\Omega_i \subset \Omega_{i+1}$ and $\cup_i \Omega_i = M$. The Dirichlet drifting heat kernel of Δ_f on Ω_i , $H_i(x, y, t)$, is defined and has the properties

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_{f,y}\right) H_i(x, y, t) &= 0 \quad \text{on } \Omega_i \times \Omega_i \times (0, T) \\ \lim_{t \rightarrow 0} H_i(x, y, t) &= \delta_x(y) e^{f(x)}. \end{aligned}$$

Furthermore, $H_i(x, y, t) > 0$ since Δ_f is self-adjoint and positive definite on $\{\Omega_i\}$. Letting $H_f(x, y, t)$ be drifting heat kernel on M , the maximum principle for the drifting heat equation implies that $H_i(x, y, t) \nearrow H_f(x, y, t)$ and that H_f is positive. Furthermore $\lambda_1(\Omega_i) \rightarrow \lambda_1(M)$.

We will prove the following result

LEMMA 2.6. *Let*

$$u(x, t) = \int_M H_f(x, y, t) u_o(y) d\mu(y)$$

be a solution to the drifting heat equation defined on $M \times [0, \infty)$.

If $g(x, t)$ satisfies

$$|\nabla g|^2 + g_t \leq 0 \quad \text{on } M \times (0, \infty),$$

then

$$F(t) = e^{2\lambda_1(M)t} \int_M e^{2g(x,t)} u^2(x, t) d\mu(x)$$

is nonincreasing for $t \in (0, \infty)$.

PROOF. For $\{\Omega_i\}$, an exhaustion of M with compact subsets, and $H_i(x, y, t)$ the Dirichlet heat kernel of Δ_f on Ω_i , define

$$u_i(x, t) = \int_{\Omega_i} H_i(x, y, t) u_o(y) d\mu(y)$$

and let $\lambda_1(\Omega_i) > 0$ be the first eigenvalue of the spectrum of Δ_f for the Dirichlet problem. $u_i(x, t)$ is now a nonnegative subsolution to the drifting heat equation with Dirichlet boundary conditions on Ω_i such that

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_f\right)u_i &\leq 0 \quad \text{on } \Omega_i \times (0, T) \\ u_i(x, t) &= 0 \quad \text{on } \partial\Omega_i \times (0, T). \end{aligned}$$

We will first show that the function

$$F_i(t) = e^{2\lambda_1(\Omega_i)t} \int_{\Omega_i} e^{2g(x,t)} u_i^2(x, t) d\mu(x)$$

is nonincreasing for $t \in (0, T)$. From the variational principle,

$$\begin{aligned} \lambda_1(\Omega_i) \int_{\Omega_i} e^{2g} u_i^2 d\mu &\leq \int_{\Omega_i} |\nabla(e^g u_i)|^2 d\mu \\ &= \int_{\Omega_i} |\nabla e^g|^2 u_i^2 d\mu - \int_{\Omega_i} e^{2g} u_i \Delta_f u_i d\mu \\ &\leq \int_{\Omega_i} |\nabla e^g|^2 u_i^2 d\mu - \int_{\Omega_i} e^{2g} u_i (u_i)_t d\mu \end{aligned}$$

after integration by parts and using the fact that u_i is a nonnegative subsolution. At the same time

$$\begin{aligned} \frac{\partial}{\partial t} \left(\int_{\Omega_i} e^{2g} u_i^2 d\mu \right) &= 2 \int_{\Omega_i} e^{2g} g_t u_i^2 d\mu + 2 \int_{\Omega_i} e^{2g} u_i (u_i)_t d\mu \\ &\leq -2 \int_{\Omega_i} e^{2g} |\nabla g|^2 u_i^2 d\mu + 2 \int_{\Omega_i} e^{2g} u_i (u_i)_t d\mu. \end{aligned}$$

Combining the two we get

$$\frac{\partial}{\partial t} \left(e^{2\lambda_1(\Omega_i)t} \int_{\Omega_i} e^{2g(x,t)} u_i^2(x, t) d\mu(x) \right) \leq 0$$

which proves the claim for F_i .

The lemma follows by letting $i \rightarrow \infty$, since $\lambda_1(\Omega_i) \rightarrow \lambda_1(M)$ and $u_i \rightarrow u$. \square

The next lemma extends the analogous result for the heat kernel of the Laplacian on a Riemannian manifold to the drifting heat kernel on a weighted manifold.

LEMMA 2.7. *Let B_1, B_2 be two bounded subsets of M with volume V_1 and V_2 respectively with respect to the measure $d\mu$. Then*

$$\int_{B_1} \int_{B_2} H_f(x, y, t) d\mu(y) d\mu(x) \leq e^{-\lambda_1(M)t} V_1^{1/2} V_2^{1/2} e^{-\mathbf{d}^2(B_1, B_2)/4t}$$

for all $t > 0$, where $\mathbf{d}(B_1, B_2)$ denotes the distance between the two sets.

PROOF. Define

$$u_i(x, t) = \int_{B_i} H_f(x, y, t) d\mu(y) \quad \text{and} \quad g_i(x, t) = \frac{\mathbf{d}^2(x, B_i)}{4(t + \epsilon)}$$

where $\mathbf{d}(x, B_i)$ is the distance between x and the set B_i , and $\epsilon > 0$ is constant. Then,

$$|\nabla g_i|^2 = \frac{|\mathbf{d} \nabla \mathbf{d}|^2}{4(t+\epsilon)^2} \leq \frac{\mathbf{d}^2(x, B_i)}{4(t+\epsilon)^2} \quad \text{and} \quad \frac{\partial g_i}{\partial t} = -\frac{\mathbf{d}^2(x, B_i)}{4(t+\epsilon)^2}.$$

Therefore, we may apply Lemma 2.6 to each pair u_i, g_i to get that

$$\int_M e^{2g_i(x,t)} u_i^2(x,t) d\mu(x) \leq e^{-2\lambda_1(M)t} \int_M e^{2g_i(x,0)} u_i^2(x,0) d\mu(x)$$

From the definition of the drifting heat kernel we know that

$$u_i(x,0) = \int_{B_i} H_f(x,y,0) d\mu(y) = \chi(B_i)$$

where $\chi(B_i)$ is the characteristic function of B_i .

Given that $g_i(x,0) = 0$ for all $x \in B_i$, we obtain

$$\int_M e^{2g_i(x,0)} u_i^2(x,0) d\mu(x) = V_i$$

hence,

$$\int_M e^{2g_i(x,t)} u_i^2(x,t) d\mu(x) \leq e^{-2\lambda_1(M)t} V_i.$$

From the triangle inequality,

$$\mathbf{d}^2(B_1, B_2) \leq [\mathbf{d}(x, B_1) + \mathbf{d}(x, B_2)] \leq 2\mathbf{d}^2(x, B_1) + 2\mathbf{d}^2(x, B_2)$$

for any $x \in M$. Therefore,

$$\begin{aligned} & \int_M e^{\mathbf{d}^2(B_1, B_2)/8(t+\epsilon)} u_1(x) u_2(x) d\mu(x) \\ (3) \quad & \leq \int_M e^{\mathbf{d}^2(x, B_1)/4(t+\epsilon)} u_1(x) e^{\mathbf{d}^2(x, B_2)/4(t+\epsilon)} u_2(x) d\mu(x) \\ & \leq \left(\int_M e^{2g_1(x,t)} u_1^2(x) d\mu(x) \right)^{1/2} \left(\int_M e^{2g_2(x,t)} u_2^2(x) d\mu(x) \right)^{1/2} \\ & \leq V_1^{1/2} V_2^{1/2} e^{-2\lambda_1(M)t}. \end{aligned}$$

On the other hand, the left side of the inequality can be rewritten as

$$\begin{aligned} & \int_M e^{\mathbf{d}^2(B_1, B_2)/8(t+\epsilon)} u_1(x) u_2(x) d\mu(x) \\ & = e^{\mathbf{d}^2(B_1, B_2)/8(t+\epsilon)} \int_M \int_{B_1} \int_{B_2} H_f(x,y,t) H_f(x,z,t) d\mu(y) d\mu(z) d\mu(x) \\ & = e^{\mathbf{d}^2(B_1, B_2)/8(t+\epsilon)} \int_{B_1} \int_{B_2} H_f(z,y,2t) d\mu(y) d\mu(z) \end{aligned}$$

from the semigroup property of the heat operator.

The lemma follows by combining this with (3) and sending $\epsilon \rightarrow 0$. \square

We are now ready to prove the Gaussian upper bounds for the drifting heat kernel.

THEOREM 2.8. *Let $(M^n, g, d\mu)$ be a metric measure space that satisfies the property (V_R) for all $x \in B_{x_0}(R)$. Let $H_f(x, y, t)$ denote the minimal drifting*

heat kernel defined on $M \times M \times (0, \infty)$ Then for any $\varepsilon > 0$ there exist constants $c_1(n, \varepsilon), c_2(n)$ such that

$$H_f(x, y, t) \leq \frac{c_1 e^{c_2 A(R)}}{V_f^{1/2}(x, \sqrt{t}) V_f^{1/2}(y, \sqrt{t})} e^{-\lambda_1 t} \cdot e^{-\frac{d^2(x, y)}{4(1+\varepsilon)t}}$$

for any $x, y \in B_{x_o}(R/2)$ and $0 < t < R^2/4$, where $\lambda_1 = \lambda_1(M)$.

The proof follows by combining Lemma 2.7 and the estimate of Lemma 2.4 (see [5] and [13, Theorem 1.1])

3. L^1 Liouville Property

For the proof of the L^1 Liouville theorem we will need that the weighted manifold is stochastically complete, in other words that

$$\int_M H_f(x, y, t) d\mu = 1.$$

Grigor'yan shows in [3, Theorem 3.13] that a sufficient condition for stochastic completeness is that

$$\int_1^\infty \frac{r}{\log V_f(x_o, r)} dr = \infty$$

for some point $x_o \in M$. In other words, it requires that the weighted volume of the manifold grows at most exponentially quadratic in r . We summarize this below

LEMMA 3.1. *Let $(M^n, g, d\mu)$ be a smooth metric measure space on which the heat kernel, $H_f(x, y, t)$, is well defined. Suppose that*

$$V_f(x_o, R) \leq C R^\alpha e^{cR^2}$$

with respect to a fixed point x_o , for all $R > 1$ and uniform constants C, c, α . Then $(M^n, g, d\mu)$ is stochastically complete.

The above result essentially dictates what the assumptions on the volume growth and Ric_f should be on our manifold; they indicate that in both the volume and heat kernel estimates as well the mean value inequalities of the previous section, the exponential term in R (in other words $A(R)$) should be at most quadratic in R . As we will see in the following section, the assumption in the theorem can be achieved under a combination of conditions for Ric_f and f .

The proof of Theorem 1.1 will be based on the arguments of [4] and as a result require the following integration by parts formula

PROPOSITION 3.1. *Under the assumptions of Theorem 1.1, if v is a nonnegative f -subharmonic function in L^1 then*

$$\int_M \Delta_{f,y} H_f(x, y, t) v(y) d\mu(y) = \int_M H_f(x, y, t) \Delta_{f,y} v(y) d\mu(y)$$

PROOF. We include an outline of the proof for the sake of completion. Stokes' Theorem on $B_{x_o}(R)$ implies that

$$\begin{aligned} (4) \quad & \left| \int_{B_{x_o}(R)} \Delta_{f,y} H_f(x, y, t) v(y) d\mu(y) - \int_{B_{x_o}(R)} H_f(x, y, t) \Delta_{f,y} v(y) d\mu(y) \right| \\ & \leq \int_{\partial B_{x_o}(R)} |\nabla H_f(x, y, t)| v(y) d\sigma_\mu(y) - \int_{\partial B_{x_o}(R)} H_f(x, y, t) |\nabla v(y)| d\sigma_\mu(y) \end{aligned}$$

where $d\sigma_\mu$ is the induced metric on $\partial B_{x_o}(R)$ by the weighted measure $d\mu$.

The goal is to show that the right side of the above inequality vanishes as $R \rightarrow \infty$. To this end, we take a cut-off function $\phi(s) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$\begin{aligned} \phi(s) &= 1 \quad \text{for } s \in [R, R+1] \\ \phi(s) &= 0 \quad \text{for } s \in [0, R-1] \cup [R+2, \infty) \\ 0 &\leq \phi(s) \leq 1 \quad \text{and} \quad |\phi'(s)| \leq \sqrt{3}. \end{aligned}$$

The mean value inequality of Corollary 2.1 implies that there exist uniform constants $C_9(n, a), C_{10}(n)$ such that all $R > 1$

$$(5) \quad \sup_{B_{x_o}(\frac{R}{2})} v \leq C_9 \frac{e^{C_{10}A(R)}}{R^2 V_f(x_o, R)} \|v\|_1.$$

Since v is nonnegative and f -subharmonic

$$\begin{aligned} 0 &\leq \int_M \phi^2 v \Delta_f v \, d\mu = - \int_M \langle \nabla(\phi^2 v), \nabla v \rangle \, d\mu \\ &\leq 2 \int_M |\nabla \phi|^2 v^2 \, d\mu - \frac{1}{2} \int_M \phi^2 |\nabla v|^2 \, d\mu \end{aligned}$$

by the generalized Cauchy's inequality. As a result,

$$\begin{aligned} \int_{B_{x_o}(R+1) \setminus B_{x_o}(R)} |\nabla v|^2 \, d\mu &\leq 4 \int_M |\nabla \phi|^2 v^2 \, d\mu \leq 12 \int_{B_{x_o}(R+2)} v^2 \, d\mu \\ &\leq \sup_{B_{x_o}(R+2)} v \cdot \|v\|_1 \leq \frac{C e^{C' R^2}}{V_f(x_o, 2R+4)} \|v\|_1^2 \end{aligned}$$

by (5), where now $C = C(n, a)$ and $C' = C'(n, b)$. Combining with the Schwartz inequality and the fact that $B_{x_o}(R+1) \setminus B_{x_o}(R) \subset B_{x_o}(2R+4)$ we get,

$$(6) \quad \begin{aligned} \int_{B_{x_o}(R+1) \setminus B_{x_o}(R)} |\nabla v| \, d\mu &\leq V_f^{1/2}(x_o, 2R+4) \left(\int_{B_{x_o}(R+1) \setminus B_{x_o}(R)} |\nabla v|^2 \, d\mu \right)^{1/2} \\ &\leq C e^{C' R^2} \|v\|_1. \end{aligned}$$

For all $x, y \in B_{x_o}(R+1)$ and $0 < t < R^2/4$ the volume comparison property of Lemma 2.2 implies that

$$V_f(x, \sqrt{t}) \leq V_f(y, \sqrt{t} + d(x, y)) \leq \left(\frac{d(x, y)}{\sqrt{t}} + 1 \right)^{a+1} e^{4bR^2} V_f(y, \sqrt{t}).$$

Therefore, for any $x \in B_{x_o}(R)$ and $0 < t < R^2/4$ the heat kernel estimate gives us

$$\begin{aligned} &\int_{B_{x_o}(R+1) \setminus B_{x_o}(R)} H_f(x, y, t) |\nabla v|(y) \, d\mu(y) \\ &\leq \left(\sup_{y \in B_{x_o}(R+1) \setminus B_{x_o}(R)} H_f(x, y, t) \right) \int_{B_{x_o}(R+1) \setminus B_{x_o}(R)} |\nabla v| \, d\mu \\ &\leq C V_f^{-1}(x, \sqrt{t}) \|v\|_1 \sup_{y \in B_{x_o}(R+1) \setminus B_{x_o}(R)} \left\{ \left(\frac{d(x, y)}{\sqrt{t}} + 1 \right)^{a+1} e^{\tilde{C} R^2 - \frac{d^2(x, y)}{5t}} \right\} \end{aligned}$$

by (6), for $C = C(n, a)$ and $\tilde{C} = \tilde{C}(n, b)$.

For x, y as above, $d(x, y) \leq R+1+d(x_o, x)$ and $d(x, y) \geq R-d(x_o, x)$. Therefore,

$$(7) \quad \begin{aligned} & \int_{B_{x_o}(R+1) \setminus B_{x_o}(R)} H_f(x, y, t) |\nabla v|(y) d\mu(y) \\ & \leq C V_f^{-1}(x, \sqrt{t}) \|v\|_1 \left(\frac{R+1+d(x_o, x)}{\sqrt{t}} + 1 \right)^{a+1} e^{\tilde{C} R^2 - \frac{(R-d(x_o, x))^2}{5t}}. \end{aligned}$$

For $T > 0$ sufficiently small and for all $t \in (0, T)$ there exists a constant $\beta > 0$ and uniform constants c, \hat{c} such that

$$\tilde{C} R^2 - \frac{(R-d(x_o, x))^2}{5t} \leq -\beta R^2 + c \frac{d^2(x_o, x)}{t} + \hat{c}.$$

As a result, as $R \rightarrow \infty$ the right side of (7) tends to zero for all $t \in (0, T)$, $x \in M$.

In a similar manner we can show that

$$(8) \quad \int_{B_{x_o}(R+1) \setminus B_{x_o}(R)} |\nabla H_f(x, y, t)| v(y) d\mu(y)$$

tends to zero as $R \rightarrow \infty$. Using the definition of ϕ , integration by parts and the generalized Cauchy's inequality we can first estimate

$$(9) \quad \begin{aligned} & \int_{B_{x_o}(R+1) \setminus B_{x_o}(R)} |\nabla H_f|^2 d\mu \leq \int_M \phi^2 |\nabla H_f|^2 d\mu \\ & \leq 4 \int_M |\nabla \phi|^2 H_f^2 d\mu - 2 \int_M \phi^2 H_f \Delta_f H_f d\mu \\ & \leq 12 \int_{B_{x_o}(R+2) \setminus B_{x_o}(R-1)} (H_f)^2 d\mu \\ & \quad + 2 \left(\int_{B_{x_o}(R+2) \setminus B_{x_o}(R-1)} (H_f)^2 d\mu \right)^{1/2} \left(\int_M (\Delta_f H_f)^2 d\mu \right)^{1/2}. \end{aligned}$$

The stochastic completeness of the weighted manifold and the Gaussian upper bound for H_f imply that for $x \in B_{x_o}(R)$

$$\begin{aligned} & \int_{B_{x_o}(R+2) \setminus B_{x_o}(R-1)} H_f(x, y, t)^2 d\mu(y) \leq \sup_{y \in B_{x_o}(R+2) \setminus B_{x_o}(R-1)} H_f(x, y, t) \\ & \leq C V_f^{-1}(x, \sqrt{t}) \left(\frac{R+2+d(x_o, x)}{\sqrt{t}} + 1 \right)^{a+1} e^{\tilde{C} R^2 - \frac{(R-1-d(x_o, x))^2}{5t}}. \end{aligned}$$

as in (7). Using the eigenfunction expansion for the heat kernel on compact domains, we can show as in [2, Lemma 7] that for some uniform constant \hat{C}

$$\int_M (\Delta_f H_f)^2 d\mu \leq \frac{\hat{C}}{t^2} H_f(x, x, t).$$

We now substitute the above two upper bounds into the right side of (9) to get

$$\begin{aligned} & \int_{B_{x_o}(R+1) \setminus B_{x_o}(R)} |\nabla H_f|^2 d\mu \leq \hat{C} \left[V_f^{-1}(x, \sqrt{t}) + \frac{1}{t} V_f^{-1/2}(x, \sqrt{t}) H_f^{1/2}(x, x, t) \right] \\ & \quad \cdot \left(\frac{R+2+d(x_o, x)}{\sqrt{t}} + 1 \right)^{a+1} e^{\tilde{C} R^2 - \frac{(R-1-d(x_o, x))^2}{10t}}. \end{aligned}$$

From the above estimate and the mean value inequality of Corollary 2.1 we see that

$$\begin{aligned}
& \int_{B_{x_o}(R+1) \setminus B_{x_o}(R)} |\nabla H_f(x, y, t)| v(y) d\mu(y) \\
& \leq \sup_{y \in B_{x_o}(R+1) \setminus B_{x_o}(R)} v(y) \cdot \int_{B_{x_o}(R+1) \setminus B_{x_o}(R)} |\nabla H_f(x, y, t)| d\mu(y) \\
& \leq \frac{e^{C(1+R^2)}}{V_f(x_o, 2R+4)} \|v\|_1^2 \cdot V_f^{1/2}(x_o, R+1) \left(\int_{B_{x_o}(R+1) \setminus B_{x_o}(R)} |\nabla H_f|^2 d\mu \right)^{1/2} \\
& \leq C V_f^{-1/2}(x_o, 2R+4) [V_f^{-1}(x, \sqrt{t}) + \frac{1}{t} V_f^{-1/2}(x, \sqrt{t}) H_f^{1/2}(x, x, t)] \\
& \quad \cdot \left(\frac{R+2+d(x_o, x)}{\sqrt{t}} + 1 \right)^{a+1} e^{\tilde{C} R^2 - \frac{(R-1-d(x_o, x))^2}{10t}}.
\end{aligned}$$

Similarly to the argument for (7) we can find a $T > 0$ sufficiently small such that for all $t \in (0, T)$ the right side of this inequality tends to zero as $R \rightarrow \infty$.

Now using the mean value theorem for integrals, we can show that the right side of (4) tends to zero as $R \rightarrow \infty$ for all $t < T$. That it tends to zero for all $t > 0$ is a consequence of the semigroup property of the heat kernel (see for example [13, Theorem 4.3]) \square

We are now ready to prove Theorem 1.1.

PROOF OF THEOREM 1.1. As in [4] we use the nonnegative f -subharmonic function v to construct a solution to the heat equation

$$v(t, x) = \int_M H_f(x, y, t) v(y) d\mu(y).$$

We observe that

$$\frac{\partial}{\partial t} v(t, x) = \int_M (\Delta_{f,y} H_f(x, y, t)) v(y) d\mu(y) = \int_M H_f(x, y, t) \Delta_{f,y} v(y) d\mu(y) \geq 0$$

by Proposition 3.1 and the subharmonicity of v . Therefore $v(t, x)$ is nondecreasing in t . At the same time,

$$\int_M v(t, x) d\mu(x) = \int_M \int_M H_f(x, y, t) v(y) d\mu(y) d\mu(x) = \int_M v(y) d\mu(y)$$

by the stochastic completeness of the weighted manifold. Given that $v(t, x)$ is nondecreasing in t , we conclude that $v(t, x) = v(x)$ for all x , and as a result $(\partial/\partial t)v(t, x) = \Delta_{f,x}v(t, x) = 0$, in other words $v(x)$ must be a nonnegative f -harmonic function.

To show that v is constant we consider the function $v_\alpha(x) := \min\{v(x), \alpha\}$ for some positive constant α . v_α is superharmonic, since v is harmonic, and it satisfies the properties

$$0 \leq v_\alpha(x) \leq v(x), \quad |\nabla v_\alpha(x)| \leq |\nabla v(x)|.$$

As a result, v_α is also in L^1 . Furthermore, it can easily be seen that v_α satisfies the mean value inequality (5): if $v_\alpha = \alpha$ then (5) clearly holds, and if not then

$$\sup_{B_{x_o}(\frac{R}{2})} v_\alpha \leq \sup_{B_{x_o}(\frac{R}{2})} v \leq C_9 \frac{e^{C_{10}A(R)}}{R^2 V_f(x_o, R)} \|v\|_1 = C'_9 \frac{e^{C_{10}A(R)}}{R^2 V_f(x_o, R)} \|v_\alpha\|_1$$

for some constant C'_9 independent of R . Using the superharmonicity of v_α we can also obtain the gradient estimate of (6). Applying a similar argument as in the proof of Proposition 3.1, we can integrate by parts to show that

$$\frac{\partial}{\partial t} \int_M H_f(x, y, t) v_\alpha(y) d\mu(y) = \int_M H_f(x, y, t) \Delta_{f,y} v_\alpha(y) d\mu(y) \leq 0$$

and obtain by the stochastic completeness of the manifold that v_α must be f -harmonic.

From the regularity of harmonic functions, it follows that v_α must satisfy either $v_\alpha \equiv v$ or $v_\alpha \equiv \alpha$. Since α was arbitrary and v is nonnegative we must have that v is a constant function. \square

4. Sufficient Conditions

In this section we will give sufficient conditions on Ric_f and f so that the manifold satisfies the assumptions of Theorem 1.1. As above we fix a point $x_o \in M$ and set $r(x) = d(x, x_o)$.

PROPOSITION 4.1. *The weighted manifold satisfies the L^1 Liouville property whenever one of the following holds:*

- (1) $\text{Ric}_f \geq -K(1 + r(x)^2)$ and $|f(x)| \leq K_o$ for some uniform constants K, K_o .
- (2) $\text{Ric}_f \geq 0$ and $|f(x)| \leq c(1 + r(x)^2)$ for some uniform constant c .
- (3) $\text{Ric}_f \geq -K(1 + r(x)^2)$ and $|\nabla f| \leq c(1 + r(x))$ for some uniform constant c .

PROOF. For (1) the proof of Theorem 1.1 in [14] shows that the manifold satisfies property (V_R) for all $R > 1$ with $a = a(n, K_o)$ and $A(R) = bR^2$ for some $b = b(n, K)$.

For (2) Lemma 2.1 in [8] shows that the manifold satisfies the property (V_R) for all $R > 1$ with $a = n - 1$ and $A(R) = 9c(1 + R^2)$.

For (3) we go back to the Laplacian comparison theorem in [14]. Let $x, y \in B_{x_o}(R)$ and γ be the minimizing geodesic from x to y such that $\gamma(0) = x$ and $\gamma(r) = y$. Whenever $\text{Ric}_f \geq -(n-1)K(R)$ on $B_{x_o}(3R)$, the Bochner formula gives us

$$\Delta_f r(x, y) \leq (n-1) \frac{\text{sn}'_K(r)}{\text{sn}_K(r)} - \frac{1}{\text{sn}_K^2(r)} \int_0^r (\text{sn}_K^2)'(t) f'(t) dt$$

where sn_K is the solution to the Riccati equation $\text{sn}''_K - K \text{sn}_K = 0$ with $\text{sn}_K(0) = 0$ and $\text{sn}'_K(0) = 1$, and $f'(t) = \langle \nabla f, \nabla r \rangle$. Observe that $\text{sn}_K(r) = \sinh(\sqrt{K}r)$ and that $(\text{sn}_K^2)' \geq 0$.

If $f'(t) \geq -c(R)$, then

$$\Delta_f r(x, y) \leq (n-1) \frac{\text{sn}'_K(r)}{\text{sn}_K(r)} + c(R) \leq \frac{n-1}{r} + \sqrt{K(R)} + c(R)$$

and integrating this inequality from r_1 to r_2 for $0 < r_1 < r_2 < R$ we get

$$\frac{J_f(x, r_2, \xi)}{J_f(x, r_1, \xi)} \leq \left(\frac{r_2}{r_1}\right)^{n-1} e^{(\sqrt{K(R)}+c(R))(r_2-r_1)} \leq \left(\frac{r_2}{r_1}\right)^{n-1} e^{(\sqrt{K(R)}+c(R))R}.$$

The proposition follows in each case by Theorem 1.1 \square

Observe that the last case reduces to assuming $\text{Ric}_f^q \geq -C(1 + r(x)^2)$ which was previously studied by X-D. Li [7].

5. Examples

The well-known examples in [6], demonstrate that for $\alpha > 2$ there exist Riemannian manifolds with $\text{Ric} \geq -c(1 + r^\alpha)$ that do not satisfy the L^1 Liouville property. Below we provide examples at the other end of the spectrum, in the sense that if $\text{Ric}_f \geq 0$ then we cannot let f have growth higher than quadratic.

EXAMPLE 5.1. Consider the euclidian plane \mathbb{R}^2 with metric $g = dr^2 + r^2 d\theta^2$ in polar coordinates, and with weighted measure $d\mu = e^{-f} dv$ where $f = f(r)$ is a function that only depends on the polar distance r . For simplicity we take $f(r) = Ar^\alpha$ with $\alpha \geq 1$ and $A \geq 0$.

Then $\text{Ric}_f = \text{Ric} + \text{Hess}f \geq 0$, and the f -Laplacian of a function u is given by

$$\Delta_f u = u_{rr} + \left(\frac{1}{r} - f_r\right)u_r + \frac{1}{r^2}u_{\theta\theta}.$$

Observe that if f is constant, then $\log r$ is a harmonic function which does not belong to L^1 , noting that the singularity at zero is not what affects the integrability.

The function

$$u(r) = \int_1^r \frac{1}{t} e^{f(t)} dt$$

is an f -harmonic function whose L^1 norm is given by

$$\begin{aligned} \|u\|_{L^1} &= 2\pi \int_0^\infty \left| \int_1^r \frac{1}{t} e^{f(t)} dt \right| r e^{-f(r)} dr \\ &= 2\pi \int_0^1 \int_r^1 \frac{r}{t} e^{f(t)-f(r)} dt dr + 2\pi \int_1^\infty \int_1^r \frac{r}{t} e^{f(t)-f(r)} dt dr. \end{aligned}$$

The first integral above is always clearly bounded since the exponential factor is bounded and $\int_0^1 \int_r^1 \frac{r}{t} dt dr = \int_0^1 r \log r dr < C$. Changing the order of integration in the second integral gives

$$2\pi \int_1^\infty \int_t^\infty \frac{r}{t} e^{f(t)-f(r)} dr dt$$

and when $\alpha = 2$ this integral diverges.

However, when $\alpha > 2$, we rewrite $r = r^{\alpha-1} t^{2-\alpha} \leq r^{\alpha-1} t^{2-\alpha}$, since $r \geq t$ on our area of integration, then

$$\begin{aligned} &2\pi \int_1^\infty \int_t^\infty \frac{r}{t} e^{f(t)-f(r)} dr dt \\ &\leq C \lim_{a,b \rightarrow \infty} \int_1^a \int_t^b r^{\alpha-1} t^{1-\alpha} e^{At^\alpha - Ar^\alpha} dr dt \\ &= C' \lim_{a,b \rightarrow \infty} \int_1^a -t^{1-\alpha} e^{At^\alpha - Ar^\alpha} \Big|_{r=t}^{r=b} dt \\ &= C' \lim_{a,b \rightarrow \infty} \int_1^a t^{1-\alpha} [1 - e^{At^\alpha - Ab^\alpha}] dt \\ &\leq C' \lim_{a,b \rightarrow \infty} \int_1^a t^{1-\alpha} dt < \infty. \end{aligned}$$

This simple example shows that it is not enough to assume $\text{Ric}_f \geq 0$ without any control on f . In particular, if f has order higher than quadratic, then there could exist an f -harmonic function with bounded L^1 norm.

EXAMPLE 5.2. Let $S = \mathbb{R} \times S^1$ be the cylinder with flat metric $g = dx^2 + d\theta^2$. Consider the smooth metric measure space $(S, g, e^{-f} dv)$ where $f = f(x)$ is a smooth concave function in x such that $f(x) = |x|^\alpha$ for all $|x| \geq R$, for some constants $R > 0$ and $\alpha \geq 2$. Then $\text{Ric}_f \geq 0$.

Observe that the function

$$u(x) = \int_0^x e^{f(t)} dt$$

is f -harmonic, since

$$\Delta_f u = u_{xx} - \nabla f \cdot \nabla u = 0.$$

We will show that u is in $L^1(d\mu)$ when $\alpha > 2$.

u is an odd function about $x = 0$, therefore

$$\|u\|_{L^1(d\mu)} = 4\pi \int_0^\infty u(x) e^{-f(x)} dx = 4\pi \int_0^\infty \int_0^x e^{f(t)-f(x)} dt dx$$

The above integral converges when $\alpha > 2$. To see this, we first change the order of integration, and then break it into two parts

$$\begin{aligned} \int_0^\infty \int_0^x e^{f(t)-f(x)} dt dx &= \int_0^\infty \int_t^\infty e^{f(t)-f(x)} dx dt \\ &= \int_0^R \int_t^\infty e^{f(t)-f(x)} dx dt + \int_R^\infty \int_t^\infty e^{f(t)-f(x)} dx dt. \end{aligned}$$

The first integral is clearly bounded, since $\int_t^\infty e^{-f(x)} dx \leq \int_0^\infty e^{-f(x)} dx < C$ for $\alpha \geq 2$, and $\int_0^R e^{f(t)} dt < C$.

For the second integral, we observe that $x \geq t$, therefore

$$\begin{aligned} \int_R^\infty \int_t^\infty e^{f(t)-f(x)} dx dt &= \int_R^\infty \int_t^\infty e^{t^\alpha - x^\alpha} dx dt \\ &\leq \int_R^\infty \int_t^\infty \frac{x^{\alpha-1}}{t^{\alpha-1}} e^{t^\alpha - x^\alpha} dx dt \\ &= \int_R^\infty \frac{-1}{(\alpha-1)t^{\alpha-1}} e^{t^\alpha - x^\alpha} \Big|_{x=t}^{x \rightarrow \infty} dt \\ &= \int_R^\infty \frac{1}{(\alpha-1)t^{\alpha-1}} dt \end{aligned}$$

since $x \geq t$. The integral is again bounded for $\alpha > 2$. Note that when $\alpha = 2$ the integral diverges.

References

- [1] D. Bakry and M. Émery, *Diffusions hypercontractives*, Séminaire de probabilités, XIX, 1983/84, Lecture Notes in Math., vol. 1123, Springer, Berlin, 1985, pp. 177–206, DOI 10.1007/BFb0075847, (to appear in print) (French). MR889476 (88j:60131)
- [2] S. Y. Cheng, P. Li, and S. T. Yau, *On the upper estimate of the heat kernel of a complete Riemannian manifold*, Amer. J. Math. **103** (1981), no. 5, 1021–1063, DOI 10.2307/2374257. MR630777 (83c:58083)
- [3] A. Grigor'yan, *Heat kernels on weighted manifolds and applications*, The ubiquitous heat kernel, Contemp. Math., vol. 398, Amer. Math. Soc., Providence, RI, 2006, pp. 93–191, DOI 10.1090/conm/398/07486, (to appear in print). MR2218016 (2007a:58028)
- [4] P. Li, *Uniqueness of L^1 solutions for the Laplace equation and the heat equation on Riemannian manifolds*, J. Differential Geom. **20** (1984), no. 2, 447–457. MR788288 (86h:58133)
- [5] ———, *Lectures on harmonic functions*. Notes.

- [6] P. Li and R. Schoen, *L^p and mean value properties of subharmonic functions on Riemannian manifolds*, Acta Math. **153** (1984), no. 3-4, 279–301, DOI 10.1007/BF02392380. MR766266 (86j:58147)
- [7] X.-D. Li, *Liouville theorems for symmetric diffusion operators on complete Riemannian manifolds*, J. Math. Pures Appl. (9) **84** (2005), no. 10, 1295–1361, DOI 10.1016/j.matpur.2005.04.002 (English, with English and French summaries). MR2170766 (2006f:58046)
- [8] O. Munteanu and J. Wang, *Smooth metric measure spaces with non-negative curvature*, Comm. Anal. Geom. **19** (2011), no. 3, 451–486. MR2843238
- [9] ———, *Analysis of weighted Laplacian and applications to Ricci solitons*, Comm. Anal. Geom. **20** (2012), no. 1, 55–94. MR2903101
- [10] L. Saloff-Coste, *Aspects of Sobolev-type inequalities*, London Mathematical Society Lecture Note Series, vol. 289, Cambridge University Press, Cambridge, 2002. MR1872526 (2003c:46048)
- [11] L. Saloff-Coste, *A note on Poincaré, Sobolev, and Harnack inequalities*, Internat. Math. Res. Notices **2** (1992), 27–38, DOI 10.1155/S1073792892000047. MR1150597 (93d:58158)
- [12] J.-Y. Wu, *L^p -Liouville theorems on complete smooth metric measure spaces with nonnegative curvature*. arXiv:1305.0616.
- [13] J.-Y. Wu and P. Wu, *Heat kernels on smooth metric measure spaces with nonnegative curvature*. arXiv:1401.6155.
- [14] G. Wei and W. Wylie, *Comparison geometry for the Bakry-Emery Ricci tensor*, J. Differential Geom. **83** (2009), no. 2, 377–405. MR2577473 (2011a:53064)

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